A non-abelian Born-Infeld Lagrangian

1981 J. Phys. A: Math. Gen. 14 3059

(http://iopscience.iop.org/0305-4470/14/11/027)

J. Phys. A: Math. Gen. **14** (1981) 3059–3065. Printed in Great Britain

A non-abelian Born–Infeld Lagrangian

T Hagiwara†
Brandeis University, Waltham, MA 02254, USA

Received 21 April 1981

Abstract. A Born–Infeld non-polynomial Lagrangian is generalised to non-abelian SU(2) gauge fields. Equations of motions are solved for a few ansätze: a plane wave solution, a chromostatic solution and a chromomagnetostatic solution.

1. Introduction

In this paper a Born–Infeld non-polynomial Lagrangian is generalised to a non-abelian SU(2) gauge theory. Classical solutions to equations of motions are obtained for several ansätze.

A Born–Infeld (BI) Lagrangian density for nonlinear but abelian electrodynamics was introduced by Born and Infeld in 1934 (Born 1934, Born and Infeld 1934) by

$$L_{\mathrm{BIED}} = M^4[(-\det(\eta_{\mu\nu}))^{1/2} - (-\det(a_{\mu\nu}))^{1/2}] \tag{1}$$

where $\eta_{\mu\nu} = (1, -1, -1, -1)$ is a Minkowski metric and

$$a_{\mu\nu} = \eta_{\mu\nu} - (1/M^2)F_{\mu\nu}. \tag{2}$$

$F_{\mu\nu}$ is a field strength tensor for an electromagnetic field A_μ: $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The Born–Infeld Lagrangian is one of the general non-derivative Lagrangians which only depend on the two algebraic Maxwell invariants $X_1 = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ and $X_2 = -\frac{1}{4}F_{\mu\nu}F^{*\mu\nu}$, but not on derivatives such as $(D_\lambda F_{\mu\nu})^2$.

The BI Lagrangian is noted amongst others for the following properties.

(i) Geometry: the BI Lagrangian density is one of the simplest non-polynomial Lagrangians that is invariant under the general coordinate transformations (with $\eta_{\mu\nu}$ replaced by a general coordinate metric $g_{\mu\nu}(x)$).

(ii) Causality: L_{BIED} is the only causal spin-1 theory (Plebanski 1968) aside from Maxwell's Lagrangian density $L = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$.

(iii) Energy density: the vacuum is characterised by $F_{\mu\nu} = 0$ as the energy density is positive semi-definite. The energy–momentum density tensor $T_{0\mu}$ satisfies the local equal-time commutation relation (Dirac 1962, Deser and Morrison 1970), that is, the Born–Infeld Lagrangian can be quantised in agreement with special relativity in spite of its nonlinearity or derivative coupling.

(iv) Helicity conservation: the maximally helicity changing $2N$-photon interaction terms ($N \geqslant 2$) vanish identically in the BI Lagrangian (Duff and Isham 1980, Hagiwara

† Work supported in part by Department of Energy under Contract DE-AC03-76ER 0320 A005.

1981). Namely, if one defines

$$(F_\pm)_{\mu\nu} = F_{\mu\nu} \pm \mathrm{i} F^*_{\mu\nu} \tag{3a}$$

$$X_\pm = 2(X_1 \pm \mathrm{i} X_2) = (\boldsymbol{E} \pm \mathrm{i}\boldsymbol{H})^2 \tag{3b}$$

then one finds that $X_\pm$ are eigenvalues for $(F^2_\pm)_{\mu\nu}$:

$$X_\pm \eta_{\mu\nu} = (F^2_\pm)_{\mu\nu}. \tag{3c}$$

The Born–Infeld Lagrangian does not contain interaction terms $(X^N_+ + X^N_-)$; $N = 2, 3, \ldots$.

(v) An intrinsic length scale and finiteness: the non-polynomial BI Lagrangian is characterised by its length scale $1/M$. The field strength $F_{\mu\nu}$ is finite everywhere while it is singular in a Maxwell theory. It approaches Coulombic at large distances, $rM \gg 1$.

(vi) Relation to supersymmetry: recently Deser and Puzalowski (1980) investigated the condition to be satisfied for the Lagrangian given in terms of $X_\pm$ alone to have a supersymmetric extension. They found that BI Lagrangians also satisfy their condition†.

With these significant features in mind it is interesting to generalise the BI Lagrangian towards non-abelian chromodynamics. In this paper we restrict our discussion within an SU(2) gauge group for simplicity.

2. Born–Infeld Lagrangian for non-abelian chromodynamics

We generalise a BI Lagrangian to non-abelian SU(2) chromodynamics. The simplest extension is given by

$$L_{\mathrm{BICD}} = M^4 [\![-\{-\mathrm{Tr}[\det(\boldsymbol{a}_{\mu\nu})]\}^{1/2} + \{-\mathrm{Tr}[\det(\eta_{\mu\nu}\mathbf{1})]\}^{1/2}]\!] / N \tag{4a}$$

where

$$\boldsymbol{a}_{\mu\nu} = \eta_{\mu\nu}\mathbf{1} - (1/M^2) F^a{}_{\mu\nu}\tau^a \tag{4b}$$

$$N = \mathrm{Tr}\,\mathbf{1} = 2 \tag{4c}$$

and τ^a is a matrix representation of an SU(2) generator. Though for simplicity we restrict our discussion within an SU(2) non-abelian chromodynamics, generalisation to a larger gauge group is straightforward. For SU(2),

$$\mathrm{Tr}(\tau^a\tau^b) = N\delta^{ab} \tag{5a}$$

$$\mathrm{Tr}(\tau^a\tau^b\tau^c\tau^d) = N(\delta^{ab}\delta^{cd} + \delta^{ad}\delta^{bc} - \delta^{ac}\delta^{bd}). \tag{5b}$$

$\boldsymbol{F}_{\mu\nu} = F_{\mu\nu}{}^a\tau^a$ is a gauge covariant field strength tensor for a non-abelian field $\boldsymbol{A}_\mu = A_\mu{}^a\tau^a$.

In the definition of a BI Lagrangian, $\det(\boldsymbol{a}_{\mu\nu})$ in (μ, ν)-space is defined uniquely by

$$\det(\boldsymbol{a}_{\mu\nu}) = \frac{1}{4!}\varepsilon_{\mu\nu\rho\sigma}\varepsilon^{\alpha\beta\gamma\delta}\boldsymbol{a}_\mu{}^\alpha\boldsymbol{a}_\nu{}^\beta\boldsymbol{a}_\rho{}^\gamma\boldsymbol{a}_\sigma{}^\delta \tag{6}$$

while Tr is for chromo-space specified by indices $a, b, \ldots$.

† Yet there may be another reason we have to consider. Namely, it may be possible, though unlikely, to interpret a non-polynomial Lagrangian as an effective Lagrangian derived from a certain renormalisable Lagrangian with heavy particles integrated over, and to determine the dimensional parameter in terms of the heavy particle masses (Hagiwara 1980).

For an SU(2) gauge group there are nine gauge invariants (Roskies 1977) among which we need only the following three in order to write a non-abelian BI Lagrangian.

$$K = -\tfrac{1}{8}\mathrm{Tr}(\boldsymbol{F}_{\mu\nu}\boldsymbol{F}^{\mu\nu}) = -\tfrac{1}{4}F^{a}{}_{\mu\nu}F^{a\mu\nu} \qquad (7a)$$

$$G = -\tfrac{1}{8}\mathrm{Tr}(\boldsymbol{F}^{\mu\nu}\boldsymbol{F}^{*}_{\mu\nu}) = -\tfrac{1}{4}F^{a}{}_{\mu\nu}F^{*a\mu\nu} \qquad (7b)$$

$$G_S{}^2 = \sum_{a,b} G_S{}^{ab}G_S{}^{ab} \qquad (7c)$$

where

$$G_S{}^{ab} = G_S{}^{ba} = -\tfrac{1}{8}[\boldsymbol{F}^{a}{}_{\mu\nu}\boldsymbol{F}^{*b\mu\nu} + \boldsymbol{F}^{b}{}_{\mu\nu}\boldsymbol{F}^{*a\mu\nu}] \qquad (7d)$$

$$\mathrm{Tr}(G_S{}^{ab}) = G. \qquad (7e)$$

In terms of K, G and G_S the BI Lagrangian is given as

$$L_{\mathrm{BICD}} = M^4\left[1 - \left(1 - \frac{2}{M^4}K - \frac{1}{3M^8}(G^2 + 2G_S{}^2)\right)^{1/2}\right]. \qquad (8)$$

Note that, in an abelian case, $G_S{}^{ab} = G$ and

$$L_{\mathrm{BIED}} = M^4\left[1 - \left(1 - \frac{2}{M^4}K - \frac{1}{M^8}G^2\right)^{1/2}\right]. \qquad (9)$$

Notice that an abelian BI Lagrangian is specified by two invariants of electrodynamics, K (or X_1) and G (or X_2), while a non-abelian Lagrangian by three. This shows a remarkable simplification in our definition of the Lagrangian, since one may expect that a non-polynomial Lagrangian is in general a function of all the nine invariants.

There is a two-fold reason behind this significant simplification. Firstly, the colour neutral Lagrangian density is derived by the trace operation instead of determinant operations in colour space. Indeed the non-polynomial Lagrangian defined by

$$L_{\mathrm{BI}} = M^4\{-[-\det_{ab}\det_{\mu\nu}(\boldsymbol{a}_{\mu\nu})]^{1/2} + [-\det_{\mu\nu}(\eta_{\mu\nu}\mathbf{1})]^{1/2}\} \qquad (10)$$

could be an equally possible non-abelian extension of the Born–Infeld Lagrangian, equation (1). All invariants appear in this definition. However, the inside of the first square root contains terms like $(F_{\mu\nu}F^{\mu\nu}$ or $F_{\mu\nu}F^{*\mu\nu})^{2N}$ for an SU(N) non-abelian gauge group and the Lagrangian expressed by invariants does not have a unique common expression independent of N.

Secondly, the definition of determinant operation equation (6) for colour matrices is completely symmetrised, thus eliminating several antisymmetric invariants such as

$$\varepsilon^{abc}F^{a}{}_{\mu\nu}F^{b}{}_{\nu\rho}F^{c}{}_{\rho\mu}. \qquad (11)$$

The Euler–Lagrange equation is given by

$$(D^{\mu}P_{\mu\nu})^{a} = (\partial^{\mu}\delta^{ac} + g\varepsilon^{abc}A^{b\mu})P^{c}{}_{\mu\nu} = 0 \qquad (12)$$

where

$$P^{a}{}_{\mu\nu} = \delta L/\delta F^{a\mu\nu} = \frac{F^{a}{}_{\mu\nu} + \tfrac{1}{3}M^{-4}F^{*b}{}_{\mu\nu}[G\delta^{ab} + 2G_S{}^{ab}]}{(1 - 2M^{-4}K - \tfrac{1}{3}M^{-8}[G^2 + 2G_S{}^2])^{1/2}}. \qquad (13)$$

$P_{\mu\nu}$ is a non-abelian extension of the dielectric displacement-like tensor in the abelian BI nonlinear electrodynamics. For example, the dichromatic constant of the space can be defined as the coefficient of $F_{\mu\nu}$ which is a function of both $F_{\mu\nu}$ and $F^*_{\mu\nu}$. Again it is trivial to show that energy density is positive semi-definite and vacuum is characterised by $F_{\mu\nu}=0$.

Equation (12) can be also written as

$$(D^\mu F_{\mu\nu})^a = j_\nu{}^a \tag{14}$$

where

$$j^{a\nu} = -\left(\frac{\partial L}{\partial K}\right)^{-1}\left[\frac{\partial K}{\partial F^a{}_{\mu\nu}}\partial\mu\left(\frac{\partial L}{\partial K}\right)+\frac{\partial G}{\partial F^a{}_{\mu\nu}}\partial\mu\left(\frac{\partial L}{\partial G}\right)\right.$$
$$\left.+\frac{\partial G_S{}^{cd}}{\partial F^a{}_{\mu\nu}}\partial\mu\left(\frac{\partial L}{\partial G_S{}^{cd}}\right)+\frac{\partial L}{\partial G_S{}^{cd}}(D\mu)^{ab}\left(\frac{\partial G_S{}^{cd}}{\partial F^b{}_{\mu\nu}}\right)\right]. \tag{15}$$

That is to say, the space chromatic-charge distribution $j_\mu{}^a(x)$ is covariantly conserved where the distribution is not given by external matter fields, but is determined by gauge fields themselves. Note that this distribution has nothing to do with the Noether symmetry current and the charge $\int d^3x\, j_0{}^a(x)$ does not correspond to any group generating charge.

3. Solutions of Born–Infeld field equations

Several trivial solutions for a set of Euler–Lagrange equations (equation (13)) are easily obtained.

(i) A plane wave solution. The ansatz for a plane wave solution propagating in a z direction is given by

$$A^a{}_{1,2}=0$$
$$A^a{}_3 = -A^a{}_0 = x_1 f^a(x_3-x_0)+x_2 g^a(x_3-x_0) \tag{16}$$

with $\partial_0 A^a{}_0 - \partial_3 A^a{}_3 = 0$. Then one finds

$$F^a{}_{01}=F^a{}_{13}=-F^{*a}{}_{01}=-F^{*a}{}_{23}=f^a(X_3-X_0)$$
$$F^a{}_{02}=F^a{}_{23}=F^{*a}{}_{01}=F^{*a}{}_{13}=g^a(X_3-X_0) \tag{17}$$
$$F^a{}_{03}=F^a{}_{12}=F^{*a}{}_{03}=F^{*a}{}_{12}=0.$$

Note that this plane-wave solution is also a solution for a non-abelian chromodynamics of an ordinary Maxwell theory first obtained by Coleman (1977). Note that, for the plane wave solution,

$$K=G=G_S{}^{ab}=0. \tag{18}$$

Also the displacement-like tensor is nothing but the field strength tensor:

$$P^a{}_{\mu\nu}=F^a{}_{\mu\nu}. \tag{19}$$

It is trivial to show that $D^\mu P_{\mu\nu}=D^\mu F_{\mu\nu}=0$.

(ii) A chromostatic solution.

$$A^a{}_i=0 \qquad A^a{}_0=\phi(r)\delta_{a3} \tag{20}$$

is the ansatz for a chromostatic solution.

$$F^a{}_{0i} = -\delta_{a3}\partial_i\phi(r) \tag{21a}$$

$$F^a{}_{jk} = F^{*a}{}_{0i} = 0 \tag{21b}$$

$$F^{*a}{}_{jk} = -\varepsilon_{ijk}F^a{}_{0i} = \varepsilon_{ijk}\delta_{a3}\partial_i\phi(r) \tag{21c}$$

$$D^\mu F_{\mu 0} = -\partial_i F^a{}_{i0} = \delta_{a3}\partial^2\phi(r). \tag{21d}$$

For Maxwellian chromodynamics the equation of motion is satisfied if $\phi(r)$ is a harmonic function (Ikeda and Miyachi 1962).

$$D^\mu F_{\mu 0} = \delta_{a3}\partial^2\phi(r) = 0 \tag{22a}$$

$$\phi(r) \sim e/r + c. \tag{22b}$$

Note that $\phi(r)$ is singular at $r = 0$.

Yet for a Born–Infeld chromodynamics, the equation of motion is given by $D^\mu P_{\mu\nu} = 0$ or $(D^\mu F_{\mu\nu})^a = j_\nu{}^a$ where the space chromatic charge distribution $j^a{}_\nu$ is not zero and is a function of $F^a{}_{\mu\nu}$ themselves. We expect that $\phi(r)$ is finite everywhere in the BI Lagrangian system. To see this we follow the example given by Born in his original paper (Born 1934).

Because

$$K = -\tfrac{1}{4}F_{\mu\nu}{}^a F^{a\mu\nu} = \tfrac{1}{2}(\partial_j\phi(r))^2 \tag{23a}$$

$$G = G_S{}^{ab} = 0 \tag{23b}$$

one finds

$$P_{0j}{}^a = -\delta_{a3}\partial_j\phi(r)/[1 + M^{-4}(\partial_k\phi(r))^2]^{1/2} \tag{24}$$

or in polar coordinates

$$P_r{}^a = -\delta_{a3}\phi'/[1 + M^{-4}(\phi')^2]^{1/2} \tag{25}$$

where $\phi' = \mathrm{d}/\mathrm{d}r\,\phi(r)$

$$(P_r{}^a)^2 = \sum_{k=1}^{3}(P^a{}_{0k})^2. \tag{26}$$

The equation of motion

$$(D^\mu P_{\mu 0})^a = -\partial_j P^a{}_{j0} = 0 \tag{27}$$

leads to a solution

$$P_r{}^a = \delta_{a3}e/r^2. \tag{28}$$

$\phi(r)$ can now be solved from equation (27) and

$$\phi' = (e/r^2)/[1 + c^2/M^4r^4]^{1/2} = (e/r_0^2)[1 + (r/r_0)^4]^{-1/2} \tag{29}$$

where $r_0 = (e/M^2)^{1/2}$. One can integrate equation (29) to obtain

$$\phi(r) = (e/r_0)f(r/r_0) \tag{30}$$

$$f(r/r_0) = -\int_{r/r_0}^{\infty} \mathrm{d}y(1 + y^4)^{-1/2} \tag{31}$$

$$= f(0) - \frac{1}{2}\int_0^{\bar\beta} \frac{\mathrm{d}\beta}{(1 - \frac{1}{2}\sin^2\beta)^{1/2}}$$

with $\bar{\beta} = 2\tan^{-1}(r/r_0)$. Note

$$f(0) = 1.8541\ldots \tag{32a}$$

$$f(x) + f(1/x) = f(0) \tag{32b}$$

$$\phi(r) \to e/r \qquad \text{as } r \to \infty. \tag{33}$$

Although $P_r^{a=3} = e/r^2$ behaves exactly as in chromatic Maxwell equations, $E_r^{a=3} = \Sigma_{i=1}^3 (F^{a=3}{}_{0i})^2 = -\mathrm{d}/\mathrm{d}r\,\phi(r)$ replaces Coulombic law but it approaches Coulombic as $r \to \infty$. It is finite everywhere and it has a discontinuity at the origin $r = 0$.

(iii) A static chromomagnetic solution. The ansatz is given in polar coordinates by

$$A^a{}_0 = 0 \qquad A^{a=1,2}{}_i = 0 \tag{34a}$$

$$\hat{A}^{a=3} = (-g/r)\tan\theta/2\hat{\phi}. \tag{34b}$$

Note that this ansatz for the vector potential provides a magnetic monopole in a string gauge (Wu and Yang 1968, Actor 1979). Polar coordinates are chosen for simplicity with $(\hat{r}, \hat{\theta}, \hat{\phi})$ as unit vectors.

$$F_{0i}{}^a = F^{*a}{}_{jk} = 0 \tag{35a}$$

$$F^{a=1,2}{}_{jk} = F^{*a=1,2}{}_{0i} = 0 \tag{35b}$$

$$\hat{B}^{a=3} = \operatorname{rot}\hat{A}^{a=3}$$

$$= (g/r^2)\hat{r} \tag{35c}$$

$$\operatorname{div}\hat{A}^{a=3} = 0. \tag{35d}$$

Equations of motion are written for $B_i = \frac{1}{2}\varepsilon_{ijk}F^{a=3}{}_{jk}$ and $H_i = \frac{1}{2}\varepsilon_{ijk}P^{a=3}{}_{jk}$ as

$$\operatorname{div}\hat{B} = g \qquad \operatorname{rot}\hat{H} = 0$$

where

$$L = 1 - (1 + B^2)^{1/2}$$

$$\hat{H} = M^4 \frac{\delta L}{\delta B} = \frac{\hat{B}}{(1 + B^2/M^4)^{1/2}}.$$

By taking an analogy to the previous section for a chromostatic solution, one can easily find a magneto-Coulombic solution,

$$B^{a=3}{}_r = g/r^2$$

while

$$H_r^{a=3} = (g/r_0^2)[1 + (r/r_0)^4]^{-1/2}$$

is finite everywhere and approaches Coulombic as $r \to \infty$. Here the cut-off length is given by $r_0 = (g/M^2)^{1/2}$.

Acknowledgments

The author is grateful to Stanley Deser; without his encouragement and numerous suggestions this paper would never have been written. He is also thankful to Rainer Puzalowski for discussions on a Born–Infeld Lagrangian and to Graham Shore on an effective Lagrangian approach.

References

Actor A 1979 *Rev. Mod. Phys.* **51** 461
Born M 1934 *Proc. R. Soc.* A **143** 410
Born M and Infeld L 1934 *Proc. R. Soc.* A **144** 425
Coleman S 1977 *Phys. Lett.* **70B** 59
Deser S and Morrison L K 1970 *J. Math. Phys.* **14** 596
Deser S and Puzalowski R 1980 *J. Phys. A: Math. Gen.* **13** 2501
Dirac P A M 1962 *Rev. Mod. Phys.* **34** 592
Duff M and Isham C J 1980 *Nucl. Phys.* B **162** 271
Hagiwara T 1981 *Nucl. Phys.* B to be published
Ikeda M and Miyachi Y 1962 *Prog. Theor. Phys.* **27** 424
Plebanski J 1968 *Lectures on Non-Linear Electrodynamics, NORDITA-Lecture Notes*
Roskies R 1977 *Phys. Rev.* D **15** 1722
Wu T T and Yang C N 1968 in *Properties of Matter under Unusual Conditions* ed H Mark and S Fernback (New York: Interscience)